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PROXIMITY INEQUALITIES AND BOUNDS FOR THE DEGREE OF INVARIANT CURVES BY FOLIATIONS OF $\mathbb{P}^2_{\mathbb{C}}$

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Abstract. In this paper we prove that if C is a reduced curve which is invariant by a foliation $\mathcal{F}$ in the complex projective plane then one has $\partial^{\underline{o}}C \leq \partial^{\underline{o}}\mathcal{F}+2+a$ where a is an integer obtained from a concrete problem of imposing singularities to projective plane curves. If $\mathcal{F}$ is nondicritical or if C has only nodes as singularities, then one gets $a = 0$ and we recover known bounds. We also prove proximity formulae for foliations and we use these formulae to give relations between local invariants of the curve and the foliation.

0. Introduction

Let $\mathcal{F}$ be a holomorphic singular foliation by curves of the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$. We will denote by $S(\mathcal{F})$ the set of singularities of $\mathcal{F}$ and we will always suppose that $S(\mathcal{F})$ is a finite set. Moreover, we will suppose that there exists an algebraic curve C of $\mathbb{P}^2_{\mathbb{C}}$ which is invariant by $\mathcal{F}$ (geometrically, this means that $C - S(\mathcal{F})$ is a leaf of $\mathcal{F}$). Already Poincaré in [13] studied the problem of bounding the degree of C. Recently the second author [5] has proved that $\partial^{\text{o}} C \leq \partial^{\text{o}}\mathcal{F} + 2$ when there are no dicritical singularities of the foliation on the curve. The same inequality had been showed by D. Cerveau and A. Lins Neto [7] for any $\mathcal{F}$ vhen all the singularities of the invariant curve are nodal. However, a bound of the degree of C only in terms of the degree of the foliation cannot be expected in the general case. To see that, we consider the family of foliations $\{\mathcal{F}_p\}_{p\in\mathbb{N}}$ where $\mathcal{F}_p$ is given by the homogeneous differential form $\omega_p = p\cdot Z\cdot Y\cdot dX - Z\cdot X\cdot dY + (1-p)\cdot Y\cdot X\cdot dZ$. Then $\mathcal{F}_p$ has degree 1 but the algebraic curve given by $X^p - Y\cdot Z^{p-1}$ is invariant by $\mathcal{F}_p$ and has degree p.

In this paper we prove that if C is a reduced curve which is invariant by a foliation $\mathcal{F}$ in $\mathbb{P}^2_{\mathbb{C}}$ then one has

$$\partial^{\text{o}} C \leq \partial^{\text{o}}\mathcal{F} + 2 + a$$

where a is an integer obtained from a concrete problem of imposing singularities to projective plane curves. More precisely, assume that $S(\mathcal{F}) = \{p_1, \ldots, p_t\}$ is the singular set of $\mathcal{F}$, A is the set of all points infinitely near the points in $S(\mathcal{F})$, and $\ell : A \to \mathbb{Z}$ a map with finite support such that

$$\nu_q(C) + s_q(\mathcal{F}) \leq \nu_q(\mathcal{F}) + 1 + \ell(q)$$

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for every $q \in A$ (where ν_q stands for the multiplicity and $s_q(\mathcal{F})$ for the number of invariant components of the exceptional divisor crossing q). Then a can be taken to be a positive integer for which there exists a polynomial of degree a satisfying all the virtual conditions imposed by ℓ [6]. If $\mathcal{F}$ is nondicritical or if C has only nodes as singularities, then one gets $a = 0$, so we recover the bounds respectively given in [5] and [7]. On the other hand, as an application, if we fix the number of tangents at the dicritical infinitely near points, or if we fix the equisingularity types of the curve at the singular points of $\mathcal{F}$, then good concrete values of a can be obtained from the fixed data, the corresponding bounds being reached in particular examples.

In the problems of imposing singularities, the proximity relations and formulae are natural tools (see Casas [6], Campillo, G.-Sprinberg, Lejeune [3] and Lipman [11],[12] for recent uses and applications). In our paper, we also prove proximity formulae for foliations and we use these formulae to give relations between local invariants of the curve and local invariants of the foliation. The proximity formulae and the proximity ideas also are the main tool and the unifying element in the paper.

1. Local preliminaries

Let S be an analytic manifold of dimension two and $\mathcal{F}$ a holomorphic singular foliation by curves of S. For fixed $p \in S$, take local coordinates $\{x, y\}$ at p and a local generator $D = a \cdot \frac{\partial}{\partial x} + b \cdot \frac{\partial}{\partial y}$, $a, b \in \mathbb{C}\{x, y\}$, of the $\mathcal{O}_{S,p}$-module $\mathcal{F}_p$. We will denote by $\nu_p(\mathcal{F})$ the minimum of $\nu(a), \nu(b)$ (where $\nu(-)$ stands for the order of power series).

Let B be an analytic branch passing through p and $\varphi\colon \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$ a primitive parametrization for B. If B is invariant by $\mathcal{F}$ then D induces a derivation $\tilde{D}\colon \mathbb{C}\{t\} \to \mathbb{C}\{t\}$ such that $\tilde{D} \circ \varphi = \varphi \circ D$. In this case write $\tilde{D} = h \cdot \frac{d}{dt}$, $h \in \mathbb{C}\{t\}$, and denote by $\mu_p(\mathcal{F}, B)$ the order $\nu(h)$, which is independent of the choice of D and φ. Now, if φ is given by the power series $\varphi_1 = \varphi(x), \varphi_2 = \varphi(y) \in \mathbb{C}\{t\}$, then

$$h \cdot \frac{d\varphi_1}{dt} = \tilde{D} \circ \varphi(x) = \varphi \circ D(x) = a(\varphi_1, \varphi_2)$$

and the same equality holds for y, so one has

$$\mu_p(\mathcal{F}, B) = \begin{cases} \nu\left(a(\varphi_1, \varphi_2)\right) - \nu(\varphi_1) + 1 & \text{if } \varphi_1 \neq 0, \\ \\ \nu\left(b(\varphi_1, \varphi_2)\right) - \nu(\varphi_2) + 1 & \text{if } \varphi_2 \neq 0. \end{cases}$$

Let us remark that $\mu_p(\mathcal{F}, B) = 0$ if and only if p is a regular point of $\mathcal{F}$. If B is not invariant by $\mathcal{F}$ denote by $\iota_p(\mathcal{F}, B)$ the order of $\varphi^*\omega$, where $\omega = b \cdot dx - a \cdot dy$, i.e.

$$\iota_p(\mathcal{F}, B) = \nu\left(b(\varphi_1, \varphi_2) \cdot \frac{d\varphi_1}{dt} - a(\varphi_1, \varphi_2) \cdot \frac{d\varphi_2}{dt}\right).$$

If p is a regular point of $\mathcal{F}$ then $\iota_p(\mathcal{F}, B) + 1$ is the intersection number at p of B and the leaf of $\mathcal{F}$ passing through p.

Let $\pi\colon \tilde{S} \to S$ be the blowup of S with center p. The blowup π is called nondicritical if the exceptional divisor $E = \pi^{-1}(p)$ of π is invariant by the strict transform $\mathcal{F}'$ of $\mathcal{F}$ by π. Otherwise, π is said to be dicritical. If B' is the strict transform of

B by π and p' is the only point in $E \cap B'$, then when B is invariant by $\mathcal{F}$ one has

$$\mu_p(\mathcal{F}, B) = \begin{cases} \mu_{p'}(\mathcal{F}', B') + \nu_p(B) \cdot \nu_p(\mathcal{F}) & \text{if } \pi \text{ is dicritical,} \\ \mu_{p'}(\mathcal{F}', B') + \nu_p(B) \cdot (\nu_p(\mathcal{F}) - 1) & \text{if } \pi \text{ is nondicritical,} \end{cases} \tag{1}$$

else

$$\iota_p(\mathcal{F}, B) = \begin{cases} \iota_{p'}(\mathcal{F}', B') + \nu_p(B) \cdot (\nu_p(\mathcal{F}) + 1) & \text{if } \pi \text{ is dicritical,} \\ \iota_{p'}(\mathcal{F}', B') + \nu_p(B) \cdot \nu_p(\mathcal{F}) & \text{if } \pi \text{ is nondicritical.} \end{cases} \tag{2}$$

Let C be a reduced algebraic curve in S. If C is invariant by $\mathcal{F}$ (i.e., if all its components are invariant) we will denote by $\mu(\mathcal{F}, C)$ the sum of $\mu_p(\mathcal{F}, B)$ for all $p \in C$ and for all branches B of C passing through p. If no one of the irreducible component of C is invariant by $\mathcal{F}$ we will denote by $\iota(\mathcal{F}, C)$ the sum of all the $\iota_p(\mathcal{F}, B)$. By looking to the local expressions of $\mathcal{F}'$ and $\tilde{S}$ it is easy to see that

$$\begin{aligned} \mu(\mathcal{F}', E) &= \nu_p(\mathcal{F}) + 1 \quad \text{if } \pi \text{ is nondicritical,} \\ \iota(\mathcal{F}', E) &= \nu_p(\mathcal{F}) - 1 \quad \text{if } \pi \text{ is dicritical.} \end{aligned} \tag{3}$$

Assume that $p \in S(\mathcal{F})$ and let λ and μ be the eigenvalues of the linear part of D. The point p is called a simple singularity of $\mathcal{F}$ [14] iff $\lambda \neq \mu \neq 0$ and $\frac{\lambda}{\mu} \notin \mathbb{Q}_+ = \{\alpha \in \mathbb{Q} \;/\; \alpha > 0\}$. Moreover, if $\lambda = 0$ then p is called a saddle-node. The simple singularities are "preserved" by blow-up: if p is so, then there are exactly two singularities in E, which are both simple; if we blow-up one of these singularities we obtain two more simple singularities, but one of them is the intersection with the strict transform of E. Thus, there exist exactly two nonsingular formal curves passing through p which are invariant by the foliation. One of these curves may be nonconvergent but both of them are convergent if p is not a saddle-node [1]. In this last case, if B_1 and B_2 are the invariant curves, one has $\mu_p(\mathcal{F}, B_i) = 1$, $i = 1, 2$.

A sequence of blowups over p is a sequence

$$S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S \tag{4}$$

where π_i is the blowup of S_{i-1} with center $p_{i-1} \in S_{i-1}$, $i = 1, \ldots, n$, $p_0 = p$ and $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_i(p_i) = p$, $i = 1, \ldots, n$. The Seidenberg's result of reduction of singularities [14] say us that there is a sequence of blowups over p such that

(a) if $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_i$ represents the strict transform of $\mathcal{F}_{i-1}$ by π_i then $p_i \in S(\mathcal{F}_i)$, $i = 0, \ldots, n-1$;
(b) each $q \in S_n$ with $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n(q) = p$ is either nonsingular or a simple singularity of $\mathcal{F}_n$.

A sequence of blowups as above is called a resolution of $\mathcal{F}$ at p.

Take a resolution of $\mathcal{F}$ at p. The point p is called dicritical if there exists $i \in \{1, 2, \ldots, n\}$ such that π_i is dicritical. Otherwise, it is called nondicritical. This condition is independent of the resolution (the blowup with center a simple singularity is nondicritical) and p nondicritical is equivalent to saying that the number of invariant branches by $\mathcal{F}$ passing through p is finite (see [4]). The foliation $\mathcal{F}$ is called a generalized curve at p iff every $q \in S_n$ with $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n(q) = p$ is not saddle-node. If $\mathcal{F}$ is locally generated at p by df for some $f \in \mathbb{C}\{x, y\}$ then p is a nondicritical singularity and $\mathcal{F}$ is a generalized curve at p (see [2]).

2. Global preliminaries

Let $\mathcal{F}$ be a holomorphic singular foliation by curves of $\mathbb{P}^2_{\mathbb{C}}$. If L is a generic straight line of $\mathbb{P}^2_{\mathbb{C}}$ (and so $L \cap S(\mathcal{F}) = \emptyset$) then the number of tangencies between L and $\mathcal{F}$, that is, points $p \in L$ such that the leaf of $\mathcal{F}$ passing through p is tangent to L, is called the degree of $\mathcal{F}$ and it will be denoted by $\partial^{\text{o}} \mathcal{F}$. The foliation $\mathcal{F}$ can be given by a homogeneous differential form

$$\Omega = \sum_{i=0}^{2} A_i(x_0, x_1, x_2) \cdot dx_i, \quad A_i \in \mathbb{C}[x_0, x_1, x_2], \quad i = 0, 1, 2,$$

on $\mathbb{C}^3$ with $\gcd(A_0, A_1, A_2) = 1$ and $R(\Omega) = 0$, where $R = x_0 \cdot \frac{\partial}{\partial x_0} + x_1 \cdot \frac{\partial}{\partial x_1} + x_2 \cdot \frac{\partial}{\partial x_2}$ is the radial vector field (that means $\sum_{i=0}^{2} x_i \cdot A_i = 0$ and it implies that $\Omega \wedge d\Omega = 0$). In the affine chart $x_2 \neq 0$ with coordinates $X = \frac{x_0}{x_2}$ and $Y = \frac{x_1}{x_2}$ the foliation is given by $\omega = A_0(X, Y, 1) \cdot dX + A_1(X, Y, 1) \cdot dY$. There is a tangency between $X = 0$ and $\mathcal{F}$ at $(0, \alpha)$ iff α is a zero of $A_1(0, Y, 1)$. Taking general coordinates, there is not a tangency at the point $x_0 = x_2 = 0$ and the zeroes of $A_1(0, Y, 1)$ are pairwise different so $\partial^{\text{o}} \mathcal{F} = \partial^{\text{o}} A_1(0, Y, 1)$. By the condition $R(\Omega) = 0$ one has $\partial^{\text{o}} A_1(0, Y, 1) = \partial^{\text{o}} A_1(x_0, x_1, x_2) - 1$ so one can conclude:

$$\partial^{\text{o}} \mathcal{F} = \partial^{\text{o}} \Omega - 1.$$

Let C be a reduced algebraic curve in $\mathbb{P}^2_{\mathbb{C}}$ and assume that C is invariant by $\mathcal{F}$. Choose the coordinates such that C cuts $(x_2 = 0)$ in exactly $\partial^{\text{o}} C$ points which are regular points of $\mathcal{F}$. The restriction to C of the vector field $A_0(X, Y, 1) \cdot \frac{\partial}{\partial Y} - A_1(X, Y, 1) \cdot \frac{\partial}{\partial X}$ extends to a meromorphic vector field D on C and D has a pole of order $\partial^{\text{o}} \mathcal{F} - 1$ at each point of $C \cap (x_2 = 0)$. Let $n \colon \tilde{C} \to C$ be the normalization of C and $\tilde{D}$ the pull-back of D by n. If for each $q \in \tilde{C}$ we denote by C_q the branch of C passing through $n(q)$ given by q then, from the definition, one has that $\mu_{n(q)}(\mathcal{F}, C_q)$ is the order of $\tilde{D}$ at the zero q. For $p \in C \cap (x_2 = 0)$ then n is an isomorphism at the neighbourhood of p, so the order of $\tilde{D}$ at the pole $n^{-1}(p)$ is $\partial^{\text{o}} \mathcal{F} - 1$ and $\mu_p(\mathcal{F}, C) = 0$. Now one can apply the theorem of Poincaré-Hopf for $\tilde{D}$ and obtain [7]:

$$\mu(\mathcal{F}, C) = \chi(\tilde{C}) + (\partial^{\text{o}} \mathcal{F} - 1) \cdot \partial^{\text{o}} C \tag{5}$$

(in the case of C reducible the above formula is nothing but the sum of the corresponding formulae for the irreducible components of C).

3. Proximity inequalities

Let S be an analytic manifold of dimension two and $p \in S$, $IN(p)$ the set of points infinitely near to p, π^p the blowup with center p and $E^p = (\pi^p)^{-1}(p)$ the exceptional divisor. If $\mathcal{F}$ is a singular holomorphic foliation by curves of S we define $\varepsilon_p(\mathcal{F})$ to be 0 if π^p is nondicritical (i.e., if E^p is invariant by the strict transform of $\mathcal{F}$) and 1 if π^p is dicritical. If $q \in IN(p)$ let $\sigma^q \colon S^q \to S$ be the composition of blowups such that $q \in S^q$, and $e_q(p)$ the number of irreducible components of the exceptional divisor $(\sigma^q)^{-1}(p)$ passing through q.

Given an analytic curve C in S', where S' is S or is obtained from S by composition of successive blowups, we will write $q \in C \cap IN(p)$ if $q \in IN(p)$, σ^q factorizes by S' locally at q and $q \in C^q$, C^q being the strict transform of C in S^q. If we speak about some invariant of C at q we are referring to the corresponding invariant of

C^q at q (for instance, $q \in C \cap IN(p)$ iff $\nu_q(C) \geq 1$). We will follow a similar rule for foliations in S'. Then, if $\mathcal{F}$ is as in the last paragraph, $\varepsilon_q(\mathcal{F})$ takes a sense and one can write $e_q(p) = s_q(\mathcal{F}, p) + s'_q(\mathcal{F}, p)$ where $s_q(\mathcal{F}, p)$ is the number of irreducible components of $(\sigma^q)^{-1}(p)$ wich are invariant by $\mathcal{F}$ (now, this expression makes sense) and $s'_q(\mathcal{F}, p)$ is the number of those ones that are not invariant.

Lemma 3.1 ([2]). *Let B be a branch passing through $p \in S$ which is not invariant by $\mathcal{F}$. Consider the infinite sequence of blowups*

$$\cdots \xrightarrow{\pi_{n+1}} S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S$$

(p_i the center of π_{i+1}, $p_0 = p$) given by the condition $p_i \in B \cap IN(p)$. Then there exists $N \in \mathbb{N}$ such that $\iota_{p_i}(\mathcal{F}, B) = 0$ for all $i \geq N$.

Proof. From (2) one has $\iota_{p_i}(\mathcal{F}, B) \geq \iota_{p_{i+1}}(\mathcal{F}, B)$, and the inequality is strict if $\nu_{p_i}(\mathcal{F}) \geq 1$ (because $\nu_{p_i}(B) > 0$ for all $i \in \mathbb{N}$), so there exists $N \in \mathbb{N}$ such that $\iota_{p_N}(\mathcal{F}, B) = \iota_{p_j}(\mathcal{F}, B)$ and $\nu_{p_j}(\mathcal{F}) = 0$ for all $j \geq N$. If $\iota_{p_N}(\mathcal{F}, B) \neq 0$ and L is the leaf passing through p_N then L and B are tangent. Then p_{N+1} is a regular point with two invariant curves passing through it (the strict transform of L and E^{p_N}) which is a contradiction. □

Definition 3.2. Given $q \in IN(p)$ we will say that q is a point proximate to p, and we denote it by $q \to p$, if $q \in E^p \cap IN(p)$. If $q \in IN(p)$ then the height of q is defined to be "the number of blowups under q", that is to say, the number of blowups that we have to do to obtain q from p. The height of q is denoted by $ht(q;p)$ or simply $ht(q)$ if it does not cause confusion. Let us remark that q is proximate to p of height 1 if and only if $q \in E^p$.

Remark 3.3. Let C be an analytic curve in S passing through p. From the classical Noether's formula (with our notation, $\nu_p(C) = \sum_{q \in E^p} I_q(E^p, C)$, where I_q represents the intersection number for curves at q) and looking at the evolution of the intersection number by blowups, we can easily obtain the proximity equality

$$\nu_p(C) = \sum_{q \to p} \nu_q(C).$$

Lemma 3.4. *Given S and $\mathcal{F}$, take a sequence of blowups over $p \in S$ as in (4). Let $E(1)$ be the exceptional divisor E^p and $E(i+1)$ the strict transform of $E(i)$ by π_i. For each $i \in \{0, \ldots, n-1\}$, let $\varepsilon_i = \varepsilon_{p_i}(\mathcal{F})$.*

(a) *If $\varepsilon_0 = 1$ then*

$$\nu_p(\mathcal{F}) - 1 = \sum_{p_i \to p} (\nu_{p_i}(\mathcal{F}_i) + \varepsilon_i) + \iota(\mathcal{F}_n, E(n)).$$

(b) *If $\varepsilon_0 = 0$ then*

$$\nu_p(\mathcal{F}) + 1 = \sum_{p_i \to p} (\nu_{p_i}(\mathcal{F}_i) + \varepsilon_i - 1) + \mu(\mathcal{F}_n, E(n)).$$

Proof. It is an easy consequence of (3) by applying (1) or (2) to each blowup. □

Proposition 3.5. *Let $\mathcal{F}$ be a foliation by curves of S and $p \in S$.*

(a) *If $\varepsilon_p(\mathcal{F}) = 1$ then*

$$\nu_p(\mathcal{F}) - 1 = \sum_{q \to p} \nu_q(\mathcal{F}) + \varepsilon_q(\mathcal{F}).$$

(b) *If* $\varepsilon_p(\mathcal{F}) = 0$ *then*

$$\nu_p(\mathcal{F}) + 1 \geq \sum_{\substack{q \to p \\ ht(q)=1}} (\nu_q(\mathcal{F}) + \varepsilon_q(\mathcal{F})) + \sum_{\substack{q \to p \\ ht(q)\geq 2}} (\nu_q(\mathcal{F}) + \varepsilon_q(\mathcal{F}) - 1)$$

and the equality holds if $\mathcal{F}$ *is a generalized curve at* p.

Proof. One can identify the set of points proximate to p with the set $E^p \times \mathbb{N}$ by doing $(q,1) = q$ and (q,i) is the only point of $E^{(q,i-1)}$ which is proximate to p when $i \geq 2$. Clearly $ht((q,i)) = i$. By simplicity, set $\nu_q(\mathcal{F}) = \nu_q$ and $\varepsilon_q(\mathcal{F}) = \varepsilon_q$ for $q \in IN(p)$.

In the case (a), since E^p is not invariant, for all $q \in E^p$, except for finitely many of them $q_1, \ldots, q_t$, one has $\iota_q(\mathcal{F}, E^p) = 0$ so, by (2) for all $i \in \mathbb{N}$ one has $\iota_{(q,i)}(\mathcal{F}, E^p) = 0$ and $\nu_{(q,i)} + \varepsilon_{(q,i)} = 0$. Moreover, by Lemma 3.1 there exists $N \in \mathbb{N}$ such that $\iota_{(q_j,N)}(\mathcal{F}, E^p) = 0$ for all $j \in \{1, \ldots, t\}$ and then $\nu_{(q_j,i)} + \varepsilon_{(q_j,i)} = 0$ when $i \geq N$. Now, we can construct a sequence of blowups like (4) with $(q_j, i) = p_{N\cdot(j-1)+i}$, $j \in \{1, \ldots, t\}$, $i \in \{1, \ldots, N\}$, and so $n = N \cdot t + 1$. Then

$$\sum_{q \to p} (\nu_q + \varepsilon_q) = \sum_{i=0}^{N} \sum_{j=1}^{t} (\nu_{(q_j,i)} + \varepsilon_{(q_j,i)}) = \nu_p - 1 - \iota(\mathcal{F}_n, E(n))$$

(this last equality by Lemma 3.4, because in this case $p_k \to p$ for all $k \in \{1, \ldots, n-1\}$). We only need to show that $\iota(\mathcal{F}_n, E(n)) = 0$, which is clear since if $x \in E(n)$ then either $x = (q,1)$, $q_j \neq q \in E^p$, or $x = (q_j, N+1)$ and in both cases $\iota_x(\mathcal{F}, E^p) = 0$.

In the case (b) let $q_1, \ldots, q_t$ be the singularities of $\mathcal{F}$ in E^p. If $q \in E^p$, $q \neq q_j$ then (q,i) is a simple point for $i \geq 2$ and then $\nu_q + \varepsilon_q = 0$ and $\nu_{(q,i)} + \varepsilon_{(q,i)} - 1 = 0$ for $i \geq 2$. By the Seidenberg theorem of reduction of singularities there exists $N \in \mathbb{N}$ such that (q_j, i) is a simple singularity for $j \in \{1, \ldots, t\}$, $i \geq N$ (since E^p is invariant by $\mathcal{F}$ if (q,k) is a regular point or a simple singularity then (q,i) is simple for all $i \geq k$) and therefore $\nu_{(q_j,i)} + \varepsilon_{(q_j,i)} - 1 = 0$. Making a sequence of blowups as in (a) and using the same arguments as above, one has:

$$\sum_{\substack{q \to p \\ ht(q)=1}} (\nu_q + \varepsilon_q) + \sum_{\substack{q \to p \\ ht(q)\geq 2}} (\nu_q + \varepsilon_q - 1)$$

$$= \sum_{j=1}^{t} \left(\nu_{(q_j,1)} + \varepsilon_{(q_j,1)} + \sum_{i=2}^{N} \nu_{(q_j,i)} + \varepsilon_{(q_j,i)} - 1 \right)$$
$$= \nu_p + 1 - \mu(\mathcal{F}_n, E(n)) + t.$$

Thus one has to prove that $\mu(\mathcal{F}_n, E(n)) \geq t$ and that the equality holds if $\mathcal{F}$ is a generalized curve at p. If $x \in E(n)$ either $x = (q,1)$, $q \neq q_j$, so x is a regular point and $\mu_x(\mathcal{F}_n, E(n)) = 0$ or $x = (q_j, N+1)$, $j \in \{1, \ldots, t\}$ so x is a simple singularity and $\mu_x(\mathcal{F}_n, E(n)) \geq 1$. It follows that

$$\mu(\mathcal{F}_n, E(n)) = \sum_{j=1}^{t} \mu_{(q_j,N+1)}(\mathcal{F}, E^p) \geq t.$$

Now, if $\mathcal{F}$ is a generalized curve at p then no one of the $(q_j, N+1)$ is a saddle-node and $\mu_{(q_j,N+1)}(\mathcal{F}, E^p) = 1$ so the equality holds. $\square$

Remark 3.6. Keeping the notations in the proof of Proposition 3.5, from the proof of (b) of this proposition it is clear that if $\mu_x(\mathcal{F}_n, E(n)) > 1$ for some $x = (q_j, N+1)$ then one has not the equality. More precisely, for each $q \in E^p$ set $r_q = \mu_{(q,N)}(\mathcal{F}, E^p) - 1$. This number is independent of N because (q, N) is a simple singularity, so $\nu_{(q,i)} = 1$ for all $i \geq N$ and then $\mu_{(q,i+1)}(\mathcal{F}, E^p) = \mu_{(q,i)}(\mathcal{F}, E^p)$. This means that $r_q = \left(\lim_{i\to\infty} \mu_{(q,i)}(\mathcal{F}, E^p)\right) - 1$. If q is a regular point then $r_q = 0$, so the set of points $q \in E^p$ with $r_q \neq 0$ is finite. The same demonstration as in (b) of Proposition 3.5 gives us

$$\nu_p + 1 = \sum_{\substack{q \to p \\ ht(q)=1}} (\nu_q + \varepsilon_q + r_q) + \sum_{\substack{q \to p \\ ht(q)\geq 2}} (\nu_q + \varepsilon_q - 1)$$

and, finally, if $\mathcal{F}$ is a generalized curve at p then $r_q = 0$ for all $q \in E^p$.

Let us give an example where the inequality (b) of Proposition 3.5 is not an equality. In a neighbourhood of $p = (0,0) \in \mathbb{C}^2$ we consider the foliation $\mathcal{F}_m$, $1 < m \in \mathbb{N}$, given by the differential form $\omega_m = y^m \cdot dx - (x \cdot y^{m-1} + x^m) \cdot dy$. There are two singularities q_1 and q_2 in E^p; q_1 is a simple singularity with $\mu_{q_1}(\mathcal{F}, E^p) = m$, so $r_{q_1} = m - 1$, and q_2 is a singularity with $\nu_{q_2} = 1$ and $\varepsilon_{q_2} = 1$. The only point in E^{q_2} proximate to p is a regular point q_3 so all the points "over q_3" which are proximate to p are simple singularities and are not saddle-node, that is to say, $r_{q_2} = 0$. The sum in all the $q \to p$ is reduced to the sum in q_1, q_2 and q_3 and we have

$$\begin{aligned}
&(\nu_{q_1} + \varepsilon_{q_1}) + (\nu_{q_2} + \varepsilon_{q_2}) + (\nu_{q_3} + \varepsilon_{q_3} - 1) \\
&\quad = (1+0) + (1+1) + (0+0-1) < m+1 \\
&\quad = \nu_p + 1
\end{aligned}$$

and the difference is, obviously, $m - 1 = r_{q_1} + r_{q_2}$.

4. Local invariants of foliations and curves

Definition 4.1. Given $q \in IN(p)$ we define $\rho_p(q)$ to be 1 if $q = p$ (i.e., if $ht(q;p) = 0$) and

$$\rho_p(q) = \sum_{q \to x} \rho_p(x)$$

if $q \neq p$. Let us remark that $q \to x$ implies $ht(x;p) < ht(q;p)$. If $q \notin IN(p)$ then we define $\rho_p(q)$ to be 0.

(This definition can be found in [2]; note that $q \to x$ means $q \in E^x \cap IN(p)$. There can be also found, with a different proof, the second part of the following lemma.)

Lemma 4.2. *If $p \in S$ and $q \in IN(p)$ then:*

(a) *$\rho_p(q)$ is the number of finite sequences of points in $IN(p)$ $q_0, q_1, \ldots, q_k$ with $q_i \to q_{i+1}$, $q_0 = q$ and $q_k = p$;*
(b) *if B is a branch passing through p such that, if x is the only point with $x \in B \cap IN(p)$ and $x \in E^q$, $e_x(p) = 1$ and $I_x(E^q, B) = 1$, then $\nu_p(B) = \rho_p(q)$.*

Proof. Part (a) is an immediate consequence from the definition of $\rho_p(q)$. We will prove the part (b) by induction on $ht(q;p)$. If $ht(q;p) = 0$ then it is trivial. If $ht(q;p) > 0$, by (a) one has

$$\rho_p(q) = \sum_{x \to p} \rho_x(q)$$

and, by the induction hypothesis, $\rho_x(q) = \nu_x(B)$ (because $ht(q;p) > ht(q;x)$ when $x \to p$) so by Remark 3.3 we are done. □

Definition 4.3. Let C be a reduced analytic curve in S and $p \in C \subset S$. Given $A \subset IN(p)$ and a map $\phi : A \to \mathbb{N}$, let us suppose that for each $q \in A$ a subset $B_q \subset E^q$ is given and let n_q be the cardinal of

$$\{\ x \in E^q \ / \ x \in C \cap IN(p) \text{ and } x \notin B_q \ \}$$

(so n_q is the number of tangent directions of C at q which do not correspond to points in B_q). We will say that C is controlled by ϕ and $\{B_q\}_{q \in A}$ iff $n_q \leq \phi(q)$ for all $q \in A$.

Lemma 4.4. *Let A be a finite subset of $IN(p)$. Given $\varphi : A \to \mathbb{Z}$, if there is a map $\ell : IN(p) \to \mathbb{Z}$ satisfying*

(a) $\ell(q) = 0$ *if* $ht(q) \gg 0$.
(b)

$$\ell(q) = \begin{cases} \sum\limits_{x \to q} \ell(x) & \textit{if } q \notin A, \\ \\ \sum\limits_{x \to q} \ell(x) + \varphi(q) & \textit{if } q \in A, \end{cases}$$

then $\ell(p) = \sum_{q \in A} \rho_p(q) \cdot \varphi(q)$.

Proof. Let $N \in \mathbb{N}$ be such that $ht(q) \geq N$ implies $\ell(q) = 0$ and $q \notin A$. We will prove that if $q \in IN(p)$ with $ht(q) \leq N$ then

$$\ell(q) = \sum_{x \in A \cap IN(q)} \rho_q(x) \cdot \varphi(x),$$

and we will do it by induction in $N - ht(q)$. If $ht(q) = N$ everything is zero and the equality holds. Now, if $ht(q) < N$ we consider two cases: $q \notin A$ and $q \in A$.

If $q \notin A$ then

$$\ell(q) = \sum_{x \to q} \ell(x) = \sum_{x \to q} \left(\sum_{y \in A \cap IN(x)} \rho_x(y) \cdot \varphi(y) \right)$$

where the second equality holds by the induction hypothesis. Since $q \notin A$ one has $\bigcup_{x \to q} A \cap IN(x) = A \cap IN(q)$ so

$$\ell(q) = \sum_{y \in A \cap IN(q)} \left(\sum_{x \to q} \rho_x(y) \right) \cdot \varphi(y) = \sum_{y \in A \cap IN(q)} \rho_q(y) \cdot \varphi(y)$$

and we are done.

If $q \in A$ then $\bigcup_{x \to q} A \cap IN(x) = A \cap IN(q) - \{q\}$ so, by the induction hypothesis, one has

$$\ell(q) = \sum_{y \in A \cap IN(q) - \{q\}} \rho_q(y) \cdot \varphi(y) + \varphi(q) = \sum_{y \in A \cap IN(q)} \rho_q(y) \cdot \varphi(y)$$

where the last equality holds because $\rho_q(q) = 1$. □

Proposition 4.5. *Let S be an analytic manifold of dimension two, $p \in S$ and $\mathcal{F}$ a foliation by curves of S. Consider the set $D = \{q \in IN(p) \ / \ \varepsilon_q(\mathcal{F}) = 1\}$ (by Seidenberg theorem it is a finite set). For each $q \in D$ let*

$$B_q(\mathcal{F}) = \{ \ x \in E^q \ / \ s_x(\mathcal{F}, p) \geq 1 \ \}$$

(since $q \in D$, $x \in B_q(\mathcal{F})$ iff x corresponds to a tangent direction of an exceptional divisor passing through q and invariant by $\mathcal{F}$). Given a map $\phi : D \to \mathbb{N}$ let C be a reduced analytic curve passing through p which is invariant by $\mathcal{F}$ and controlled by ϕ and $\{B_q(\mathcal{F})\}_{q \in D}$. Then

$$\nu_p(\mathcal{F}) + 1 + \ell_p(\phi, \mathcal{F}) \geq \nu_p(C)$$

where $\ell_p(\phi, \mathcal{F}) = \sum_{q \in D} \rho_p(q) \cdot \varphi(q)$ and $\varphi(q) = \max(0, \phi(q) + s_q(\mathcal{F}, p) - 2)$.

Proof. We will prove that for all $q \in IN(p)$ one has

$$\nu_q(\mathcal{F}) + 1 + \ell_q(\phi, \mathcal{F}) \geq \nu_q(C) + s_q(\mathcal{F}, p)$$

where $\ell_q(\phi, \mathcal{F})$ is inductively defined to be 0 if q is a simple or nonsingular point of $\mathcal{F}$ and

$$\ell_q(\phi, \mathcal{F}) = \begin{cases} \sum_{x \to q} \ell_x(\phi, \mathcal{F}) & \text{if } q \notin D, \\ \sum_{x \to q} \ell_x(\phi, \mathcal{F}) + \varphi(q) & \text{if } q \in D \end{cases}$$

otherwise. By Lemma 4.4, when $q = p$ we have the required inequality as, trivially, $s_p(\mathcal{F}, p) = 0$.

We denote by $s(q)$ the number of blowing-ups that are needed to solve the singularity of $\mathcal{F}$ at q. Note that $q \to x$ implies $s(x) > s(q)$ if $s(x) > 0$. We will use induction on $s(q)$. Let us remark that $\nu_q(C) + s_q(\mathcal{F}, p)$ is the multiplicity at q of the union of some branches invariant by $\mathcal{F}$ (corresponding to branches of the strict transform of C or of the exceptional divisor). If $s(q) = 0$, so $\ell_q(\mathcal{F}, \phi) = 0$, then either q is a regular point, so there is only one invariant curve passing through q which is nonsingular and $\nu_p(C) + s_q(\mathcal{F}, p) \leq 1 = \nu_p(\mathcal{F}) + 1$, or q is simple, so the invariant curve passing through q is nonsingular or has a nodal singularity and again $\nu_p(C) + s_q(\mathcal{F}, p) \leq 2 = \nu_p(\mathcal{F}) + 1$. If $s(q) > 0$, by the induction hypothesis, the inequality holds for all $x \to q$. We have two cases to consider, $q \notin D$ and $q \in D$.

If $q \notin D$ then

$$\begin{aligned}
\nu_q(\mathcal{F})+1 &\geq \sum_{\substack{x\to q\\ ht(x;q)=1}} (\nu_x(\mathcal{F})+\varepsilon_x(\mathcal{F})) + \sum_{\substack{x\to q\\ ht(x;q)\geq 2}} (\nu_x(\mathcal{F})+\varepsilon_x(\mathcal{F})-1)\\
&\geq \sum_{\substack{x\to q\\ ht(x;q)=1}} (\nu_x(C)+s_x(\mathcal{F},p)-\ell_x(\mathcal{F},\phi)-1+\varepsilon_x(\mathcal{F}))\\
&\quad + \sum_{\substack{x\to q\\ ht(x;q)\geq 2}} (\nu_x(C)+s_x(\mathcal{F},p)-\ell_x(\mathcal{F},\phi)-1+\varepsilon_x(\mathcal{F})-1)\\
&= \sum_{x\to q}\nu_x(C)-\sum_{x\to q}\ell_x(\mathcal{F},\phi)+K\\
&= \nu_q(C)-\ell_q(\mathcal{F},\phi)+K
\end{aligned}$$

where

$$K = \sum_{\substack{x\to q\\ ht(x;q)=1}} (\varepsilon_x(\mathcal{F})+s_x(\mathcal{F},p)-1) + \sum_{\substack{x\to q\\ ht(x;q)\geq 2}} (\varepsilon_x(\mathcal{F})+s_x(\mathcal{F},p)-2)$$

so we only need to check the equality $K = s_q(\mathcal{F},p)$. Let us remark that $x \to q$ and $q \notin D$ implies $s_p(\mathcal{F},x) \geq 1$. Now, recall the notation of the proof of Proposition 3.5 and for each $x \to q$ write $x = (y,i)$, $y \in E^q$, $i = ht(x;q)$. Then

$$K = \sum_{y\in E^q}(s_y(\mathcal{F},p)-1) + \sum_{y\in E^q}\sum_{i\geq 2}(\varepsilon_{(y,i-1)}(\mathcal{F})+s_{(y,i)}(\mathcal{F},p)-2)$$

and for each $y \in E^q$, $i \geq 2$, one has $\varepsilon_{(y,i-1)}(\mathcal{F})+s_{(y,i)}(\mathcal{F},p) = 2$ because $(q,i) \in E^{(q,i-1)}$. Finally, the number of points $y \in E^q$ such that $s_y(\mathcal{F},p)-1 = 1$ is exactly $s_q(\mathcal{F},p)$ (the points corresponding to the tangent spaces of the exceptional divisors passing through q and invariant by $\mathcal{F}$).

Now, suppose $q \in D$, and set

$$H = \{y \in E^q \ / \ \nu_y(C) \geq 1\} \cup B_q(\mathcal{F})$$

which is a finite set. One has $\#(H - B_q(\mathcal{F})) \leq \phi(q)$ as C is controlled by ϕ and $\{B_q(\mathcal{F})\}_{q\in D}$. From the definition of H, if $x \to q$ with $\nu_x(C) \neq 0$ then there are $y \in H$ and $i \in \mathbb{N}$ with $x = (y,i)$ so

$$\nu_q(C) = \sum_{x\to q}\nu_x(C) = \sum_{y\in H}\sum_{i\geq 1}\nu_{(y,i)}(C).$$

Since $\ell_x(\mathcal{F},\phi) \geq 0$ for all x one has

$$\sum_{x\to q}\ell_x(\mathcal{F},\phi) \geq \sum_{y\in H}\sum_{i\geq 1}\ell_{(y,i)}(\mathcal{F},\phi).$$

Thus, one gets

$$\begin{aligned}
\nu_q(\mathcal{F}) - 1 &= \sum_{x \to q} \nu_x(\mathcal{F}) + \varepsilon_x(\mathcal{F}) \\
&\geq \sum_{y \in H} \sum_{i \geq 1} \nu_{(y,i)}(\mathcal{F}) + \varepsilon_{(y,i)}(\mathcal{F}) \\
&\geq \sum_{y \in H} \sum_{i \geq 1} \nu_{(y,i)}(C) + s_{(y,i)}(\mathcal{F}, p) - \ell_{(y,i)}(\mathcal{F}, \phi) - 1 + \varepsilon_{(y,i)}(\mathcal{F}) \\
&\geq \nu_q(C) - \sum_{x \to q} \ell_x(\mathcal{F}, \phi) + \sum_{y \in H} \sum_{i \geq 1} s_{(y,i)}(\mathcal{F}, p) + \varepsilon_{(y,i)}(\mathcal{F}) - 1.
\end{aligned}$$

Now, $q \in D$ implies $s_{(y,i)}(\mathcal{F}, p) + \varepsilon_{(y,i-1)}(\mathcal{F}) = 1$ so

$$\sum_{y \in H} \sum_{i \geq 1} s_{(y,i)}(\mathcal{F}, p) + \varepsilon_{(y,i)}(\mathcal{F}) - 1 = \sum_{y \in H} s_y(\mathcal{F}, p) - 1 \geq -\phi(q)$$

and we can conclude

$$\nu_q(\mathcal{F}) - 1 \geq \nu_q(C) - \sum_{x \to q} \ell_x(\mathcal{F}, \phi) - \phi(q).$$

The required inequality follows immediately from the above one and the inductive definition of $\ell_q(\mathcal{F}, \phi)$. □

Corollary 4.6. *Assume that $p \in S$ is a nondicritical singularity of $\mathcal{F}$ and let C be a reduced analytic curve passing through p which is invariant by $\mathcal{F}$. Then $\nu_p(\mathcal{F}) + 1 \geq \nu_p(C)$.*

Moreover, if $\mathcal{F}$ is a generalized curve at p and C is the union of all the invariant curves passing through p then the equality holds.

Proof. Since in our case $D = \emptyset$ the first part is clear. When $\mathcal{F}$ is a generalized curve at p and C is the union of all the invariant curves then $\mathcal{F}$ is a generalized curve at q and the union of all the invariant curves passing through q is the union of the strict transform of C and the exceptional divisors passing through q, for all $q \to p$. Then we can apply induction: if q is a regular point or a simple singularity then the equality is easy; in the other cases by repeating the proof of the proposition (we only need to consider the case $q \notin D$) one sees that, now, all the inequalities are equalities (recall that in (b) of Proposition 3.5 the equality holds in the case of generalized curve). □

Remark 4.7. This corollary generalizes a classical result by Dulac [9] which says that, given S and $\mathcal{F}$ as in the proposition, if there are more than $\nu_p(\mathcal{F})+1$ invariant irreducible curves passing through $p \in S$ then there are infinitely many. Our corollary is equivalent to saying that if there is an invariant curve with multiplicity greater than $\nu_p(\mathcal{F}) + 1$ (and we can take the union of the irreducible curves) then there are an infinite number of invariant curves passing through p. In [8] Cerveau and Mattei gave a proof of the result of Dulac by induction on $s(p)$. This is possible because if p satisfies the hypothesis then there is some point in E^p satisfying also those hypotheses; however note that such a property is not clear with the hypothesis of the statement obtained from our corollary. In our case, the proximity formula allow us to use the method of induction.

The last part of Corollary 4.6 is proved in [2]. In fact, as A. Lins Neto pointed out to us, the first part can be proved too using results of [2] (with a simple generalization of the arguments given there to prove the second part).

5. Bounding the degree

Lemma 5.1. *Let $\mathcal{F}$ be a foliation by curves of an analytic two-dimensional manifold S, C a curve of S invariant by $\mathcal{F}$, $p \in C$, $f(x,y) \in \mathbb{C}\{x,y\}$ a local reduced equation of C at p, $\mathcal{G}$ the foliation of a neighbourhood of p given by df and $C\{p\}$ the set of branches of C passing through p. If $\ell : IN(p) \to \mathbb{Z}$ is a map such that $\{q \in IN(p) \ / \ \ell(q) \neq 0\}$ is a finite set and*

$$\nu_q(\mathcal{F}) + 1 + \ell(q) \geq \nu_q(C) + s_q(\mathcal{F}, p)$$

for all $q \in IN(p)$ then

$$\sum_{B \in C\{p\}} \mu_p(\mathcal{F}, B) - \sum_{B \in C\{p\}} \mu_p(\mathcal{G}, B) \geq - \sum_{q \in IN(p)} \nu_q(C) \cdot \ell(q).$$

Proof. Let $B_1, \ldots, B_t$ be the branches of C at p and p_{ij} the only point of $IN(p)$ such that $ht(p_{ij}) = j$ and $p_{ij} \in B_i$ (we can have $p_{ij} = p_{kj}$ with $k \neq i$). For the sake of simplicity let

$$\mu_i = \mu_p(\mathcal{F}, B_i) - \mu_p(\mathcal{G}, B_i)$$

and

$$\mu = \sum_{i=1}^{t} \mu_i = \sum_{B \in C\{p\}} \mu_p(\mathcal{F}, B) - \sum_{B \in C\{p\}} \mu_p(\mathcal{G}, B).$$

We recall that p is a nondicritical singularity of $\mathcal{G}$ and $\mathcal{G}$ is a generalized curve at p so it has the same property at all the points of $IN(p)$; by Corollary 4.6 we have $\nu_q(\mathcal{G}) = \nu_q(C) + e_q(p) - 1$ for all $q \in IN(p)$. After a repeated application of (1) one obtains

$$\begin{aligned} \mu_i \quad = \quad & \left(\sum_{j=1}^{N-1} \nu_{p_{ij}}(B_i) \cdot (\nu_{p_{ij}}(\mathcal{F}) - \varepsilon_{p_{ij}}(\mathcal{F}) - \nu_{p_{ij}}(C) - e_{p_{ij}}(p) + 1) \right) \\ & + \mu_{p_{iN}}(\mathcal{F}, B_i) - \mu_{p_{iN}}(\mathcal{G}, B_i). \end{aligned}$$

If $N \gg 0$ then p_{iN} is a simple singularity of $\mathcal{F}$ and $\mathcal{G}$ (from the Seidenberg theorem it is a simple singularity or a regular point, but B_i invariant and p_{ij} regular implies p_{ij+1} simple) so

$$\mu_{p_{iN}}(\mathcal{F}, B_i) \geq 1 = \mu_{p_{iN}}(\mathcal{G}, B_i)$$

($\mathcal{G}$ is a generalized curve at p). Thus, from the hypothesis assumed on ℓ, for $N \gg 0$ one has

$$\begin{aligned}
\mu &\geq \sum_{i=1}^{t}\sum_{j=1}^{N-1} \nu_{p_{ij}}(B_i)\cdot(\nu_{p_{ij}}(\mathcal{F}) - \varepsilon_{p_{ij}}(\mathcal{F}) - \nu_{p_{ij}}(C) - e_{p_{ij}}(p) + 1)\\
&\geq \sum_{i=1}^{t}\sum_{j=1}^{N-1} \nu_{p_{ij}}(B_i)\cdot(\varepsilon_{p_{ij}}(\mathcal{F}) - e_{p_{ij}}(p) + s_{p_{ij}}(\mathcal{F},p) - \ell(p_{ij}))\\
&\geq \sum_{i=1}^{t}\sum_{j=1}^{N-1} -\nu_{p_{ij}}(B_i)\cdot\ell(p_{ij})\\
&\quad + \sum_{i=1}^{t}\left(\sum_{j=1}^{N-1} \nu_{p_{ij}}(B_i)\cdot\varepsilon_{p_{ij}}(\mathcal{F}) - \nu_{p_{ij}}(B_i)\cdot s'_{p_{ij}}(\mathcal{F},p)\right).
\end{aligned}$$

We can assume, taking N greater, if it is necessary, that $\varepsilon_{p_{ij}}(\mathcal{F}) = s'_{p_{ij}}(\mathcal{F},p) = 0$ when $j \geq N$. Since $\nu_q(B_i) \neq 0$ implies $q = p_{ij}$ for some j, one has

$$\begin{aligned}
&\sum_{j=1}^{N-1} \nu_{p_{ij}}(B_i)\cdot\varepsilon_{p_{ij}}(\mathcal{F}) - \nu_{p_{ij}}(B_i)\cdot s'_{p_{ij}}(\mathcal{F},p)\\
&\qquad\qquad = \sum_{q\in IN(p)} \nu_q(B_i)\cdot\varepsilon_q(\mathcal{F}) - \nu_q(B_i)\cdot s'_q(\mathcal{F},p).
\end{aligned}$$

Now, let D be as in Proposition 4.5; that is to say, $q \in D$ iff $\varepsilon_q(\mathcal{F}) = 1$. Since $s'_x(\mathcal{F},p)$ is the number of $q \in D$ such that $x \to q$, by Remark 3.3, one has

$$\sum_{x\in IN(p)} \nu_x(B_i)\cdot s'_x(\mathcal{F},p) = \sum_{q\in D}\sum_{x\to q} \nu_x(B_i) = \sum_{q\in D}\nu_q(B_i) = \sum_{q\in IN(p)} \nu_q(B_i)\cdot\varepsilon_q(\mathcal{F}).$$

Then we are done because for $N \gg 0$ one also has $\ell(p_{ij}) = 0$ and so

$$\begin{aligned}
\mu &\geq \sum_{i=1}^{t}\sum_{j=1}^{N-1} -\nu_{p_{ij}}(B_i)\cdot\ell(p_{ij})\\
&= -\sum_{i=1}^{t}\sum_{q\in IN(p)} \nu_q(B_i)\cdot\ell(q)\\
&= -\sum_{q\in IN(p)} \nu_q(C)\cdot\ell(q).
\end{aligned}$$

□

Definition 5.2. Given $p \in S$, the blowup $\pi = \pi^p : S^p \to S$, and a number $\nu \in \mathbb{Z}$, let $C \subset S$ be a curve not necessarily reduced with $\nu_p(C) \geq \nu$. The virtual transform of C with virtual multiplicity ν is the curve

$$\tilde{C} = C^p + (\nu_p(C) - \nu)E^p;$$

that is to say, if $f \in \mathcal{O}_{S,p}$ is a generator of the ideal of C at p, p' is a point of S^p, $t \in \mathcal{O}_{S^p,p'}$ is a generator of the ideal of the exceptional divisor E^p of π at p' and $\pi^* : \mathcal{O}_{S,p} \to \mathcal{O}_{S^p,p'}$ is the map induced by π (we recall that the ideal of π^*C is generated by $\pi^*(f)$ and $t^{-\nu_p(C)}\cdot\pi^*(f)$ is a generator of the ideal of the strict transform C^p of C) then the ideal of $\tilde{C}$ at p' is generated by $t^{-\nu}\cdot\pi^*(f)$.

Now, take $\ell : IN(p) \to \mathbb{Z}$ such that $\{q \in IN(p)/\ell(q) \neq 0\}$ is a finite set. We will call height of ℓ to the maximum of $ht(q;p)$ when $q \in IN(p)$ and $\ell(q) \neq 0$. Given a curve $C \subset S$ and ℓ as above, we are going to say (by induction on $ht(\ell)$, the height of ℓ) when C goes through the points of $IN(p)$ with the virtual multiplicities given by ℓ, or, in short, C goes virtually through ℓ. When $ht(\ell) = 0$, C goes virtually through ℓ iff $\nu_p(C) \geq \ell(p)$. When $ht(\ell) > 0$ let $p_1, p_2, \ldots, p_t$ be the points of E^p with $\ell(p_i) \neq 0$ and let ℓ_i be the restriction of ℓ to $IN(p_i)$, for $i = 1, \ldots, t$; then C goes virtually through ℓ iff $\nu_p(C) \geq \ell(p)$ and the virtual transform $\tilde{C}$ of C with virtual multiplicity $\ell(p)$ goes virtually through ℓ_i for every $i \in \{1, \ldots, t\}$ (note that this makes sense because $ht(\ell_i) = ht(\ell) - 1$).

Remark 5.3. The last definition can be found in [6], and it is proved there ([6, Lemma 8.3]) that if C and D are curves of S without common components, p a point in $C \cap D$ with $\nu_p(D) \geq \nu$, and $\tilde{D}$ is the virtual transform of D with virtual multiplicity ν then one has

$$I_p(C, D) = \nu_p(C) \cdot \nu + \sum_{q \in E^p} I_q(C^p, \tilde{D}).$$

If D goes virtually through ℓ then a recursive use of this result give us

$$I_p(C, D) \geq \sum_{q \in IN(p)} \nu_q(C) \cdot \ell(q).$$

If $p_1, \ldots, p_t$ are points of $\mathbb{P}^2_{\mathbb{C}}$ and $A = \bigcup_{i=1}^t IN(p_i)$, given a map $\ell : A \to \mathbb{Z}$ such that the set $\{q \in A/\ell(q) \neq 0\}$ is finite, let ℓ_i be the restriction of ℓ to $IN(p_i)$ and let $H_\ell(a)$ be, where $a \in \mathbb{N}$, the linear system of projective plane curves of degree a such that its germ at p_i goes virtually through ℓ_i for all $i \in \{1, \ldots, t\}$. One has a coherent ideal sheaf $\mathcal{I}_\ell \subset \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}$ such that the fiber $\mathcal{I}_{\ell, p_i}$ consists of the germs at p_i going virtually through ℓ_i for $i = \{1, \ldots, t\}$ and $\mathcal{I}_{\ell,p} = \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},p}$ if $p \notin \{p_1, \ldots, p_t\}$. The discharge principle (see [6]) says that the function ℓ can be replaced by another one ℓ' such that $\mathcal{I}_\ell = \mathcal{I}_{\ell'}$ and ℓ' satisfies the following proximity inequalities:

$$\ell'(q) \geq \sum_{x \to q} \ell'(x)$$

for any $q \in A$ and $\ell'(q) \geq 0$ for every $q \in A$. Moreover, ℓ' can be computed from ℓ by an easy algorithm and ℓ' is uniquely determined by ℓ.

For each $a \in \mathbb{N}$, the global sections of the sheaf $\mathcal{I}_\ell(a) = \mathcal{I}_\ell \otimes \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(a)$ are nothing but the polynomials of degree a giving the curves in the linear system $H_\ell(a)$. By Serre's theory, the ideal sheaf $\mathcal{I}_\ell(a)$ is generated by its global sections for a large enough. In the situation of the theorem below we will only need the less restrictive assumption that $H^0\left(\mathbb{P}^2_{\mathbb{C}}, \mathcal{I}_\ell(a)\right) \neq 0$ for some a. In general, the search for such an integer a with small value depends on the position of the points p_i and $q \in IN(p_i)$ with $\ell(q) \neq 0$. Some formula for a with $H^0\left(\mathbb{P}^2_{\mathbb{C}}, \mathcal{I}_\ell(a)\right) \neq 0$ can be given. In fact, from the exact sequence

$$\begin{aligned} 0 \to H^0\left(\mathbb{P}^2_{\mathbb{C}}, \mathcal{I}_\ell(a)\right) &\to H^0\left(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(a)\right) \\ &\to H^0\left(\mathbb{P}^2_{\mathbb{C}}, \frac{\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}}{\mathcal{I}_\ell}(a)\right) \to H^1\left(\mathbb{P}^2_{\mathbb{C}}, \mathcal{I}_\ell(a)\right) \to \ldots \end{aligned}$$

it follows that

$$\dim_{\mathbb{C}} H^0\left(\mathbb{P}^2_{\mathbb{C}}, \mathcal{I}_\ell(a)\right) \geq \frac{(a+1)\cdot(a+2)}{2} - \sum_{i=1}^{t} \dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},p_i}}{\mathcal{I}_{\ell,p_i}}\right).$$

Now, from the Hoskin-Deligne formula (see [10] and [11]) one has

$$\dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},p_i}}{\mathcal{I}_{\ell,p_i}}\right) = \sum_{q\in IN(p_i)} \frac{\ell'(q)\cdot(\ell'(q)+1)}{2},$$

where ℓ' is the functor obtained from ℓ by the discharge principle (this means that the conditions imposed by ℓ' are independent on the germs at p_i). Thus, one has

$$\dim_{\mathbb{C}} H^0\left(\mathbb{P}^2_{\mathbb{C}}, \mathcal{I}_\ell(a)\right) \geq \frac{(a+1)\cdot(a+2)}{2} - \sum_{q\in A} \frac{\ell'(q)\cdot(\ell'(q)+1)}{2},$$

and $H^0\left(\mathbb{P}^2_{\mathbb{C}}, \mathcal{I}_\ell(a)\right) \neq 0$ if the right-hand side member of above inequality is positive.

Finally note that $H^1\left(\mathbb{P}^2_{\mathbb{C}}, \mathcal{I}_\ell(a)\right) = 0$ for $a \gg 0$, so for such a the conditions imposed by ℓ' on the polynomials of degree a are independent.

Theorem 1. *Let $\mathcal{F}$ be a foliation by curves of $\mathbb{P}^2_{\mathbb{C}}$, C a reduced projective plane curve invariant by $\mathcal{F}$, $A = \bigcup_{p\in S(\mathcal{F})\cap C} IN(p)$ and $\ell : A \to \mathbb{N}$ a map such that $\{q \in A/\ell(q) \neq 0\}$ is a finite set and*

$$\nu_q(\mathcal{F}) + 1 + \ell(q) \geq \nu_q(C) + s_q(\mathcal{F}, p_q)$$

for all $q \in A$, where $p_q \in S(\mathcal{F}) \cap C$ is given by $q \in IN(p_q)$. Let $a \in \mathbb{N}$ be such that the linear system $H_\ell(a)$ is nonempty. Then one has

$$\partial^{o} C \leq \partial^{o}\mathcal{F} + 2 + a.$$

Proof. Take homogeneous coordinates $\{X_0, X_1, X_2\}$ of $\mathbb{P}^2_{\mathbb{C}}$ such that $\#(C \cap L_\infty) = \partial^{o} C$ and $L_\infty \cap S(\mathcal{F}) = \emptyset$, where L_∞ is the line $X_2 = 0$. Let $F \in \mathbb{C}[X_0, X_1, X_2]$ be a reduced homogeneous equation of C; then $F(\frac{X_0}{X_2}, \frac{X_1}{X_2}, 1)$ gives us a polynomial function f from the complement of L_∞. We denote by $\mathcal{G}$ the foliation of $\mathbb{P}^2_{\mathbb{C}}$ which extends the foliation given by df. Now, by (5), since $\partial^{o}\mathcal{G} = \partial^{o} C - 1$, we have

$$\mu(\mathcal{F}, C) - \mu(\mathcal{G}, C) = \partial^{o} C \cdot (\partial^{o}\mathcal{F} + 1 - \partial^{o} C).$$

Given $p \in C$, $p \notin S(\mathcal{F})$ we have two cases. If $p \notin L_\infty$ then $\nu_p(C) = 1$ (because a curve invariant by a foliation cannot be singular at a regular point of the foliation; we remark that C is the only branch of C at p and $\mu_p(\mathcal{F}, C)$ makes sense) so $\mu_p(\mathcal{F}, C) = \mu_p(\mathcal{G}, C) = 0$. In the other case, if $C \cap L_\infty = \{p_1, p_2, \ldots, p_m\}$ (so $m = \partial^{o} C$) we claim that $\mu_{p_i}(\mathcal{G}, C) = 1$ for $i = 1, \ldots, m$ (we recall that C is regular at p_i). The curves $\{F(X_0, X_1, X_2) - \lambda \cdot X_2^m = 0\}$, where $\lambda \in \mathbb{C}$ are invariant ones by $\mathcal{G}$ so if one takes in p_i the local coordinates $\{x = \frac{X_2}{X_1}, y = F(\frac{X_0}{X_1}, 1, \frac{X_2}{X_1})\}$ (here we suppose $X_1(p_i) \neq 0$) then the curves $y - \lambda \cdot x^m = 0$ are invariant by $\mathcal{G}$. This means that $\mathcal{G}$ at p_i is given by $m \cdot y \cdot dx - x \cdot dy$ and C is given by $\{y = 0\}$ so a simple computation gives us $\mu_{p_i}(\mathcal{G}, C) = 1$. Since $p_i \notin S(\mathcal{F})$ we have $\mu_{p_i}(\mathcal{F}, C) = 0$.

Now, take $D \in H_\ell(a)$. First assume that D and C have no common component. For each $p \in C \cap S(\mathcal{F})$ let $C\{p\}$ be as in Lemma 5.1, and apply this lemma, Remark

5.3 and the Bezout theorem. Then one gets

$$\begin{aligned}
-a \cdot \overset{\circ}{\partial} C &= -\sum_{p \in C \cap S(\mathcal{F})} I_p(C, D) \\
&\leq -\sum_{p \in C \cap S(\mathcal{F})} \sum_{q \in IN(p)} \nu_q(C) \cdot \ell(q) \\
&\leq \sum_{p \in C \cap S(\mathcal{F})} \left(\sum_{B \in C\{p\}} \mu_p(\mathcal{F}, B) - \sum_{B \in C\{p\}} \mu_p(\mathcal{G}, B) \right) \\
&= \overset{\circ}{\partial} C \cdot (\overset{\circ}{\partial} \mathcal{F} + 1 - \overset{\circ}{\partial} C) + \sum_{i=1}^{m} \mu_{p_i}(\mathcal{G}, C) \\
&= \overset{\circ}{\partial} C \cdot (\overset{\circ}{\partial} \mathcal{F} + 2 - \overset{\circ}{\partial} C)
\end{aligned}$$

and the required inequality holds because $\overset{\circ}{\partial} C > 0$.

In the general case, since C is reduced, we can write $D = B + D'$ and $C = B + C'$ where D' and C' have no common component. Put $a' = \overset{\circ}{\partial} D'$ and so $\overset{\circ}{\partial} B = a - a'$ and $\overset{\circ}{\partial} C' = \overset{\circ}{\partial} C - a + a'$. Consider the map $\tilde{\ell} : A \to \mathbb{Z}$ given by $\tilde{\ell}(q) + \nu_q(B) = \ell(q)$. It is clear that C' satisfies the inequalities

$$\nu_q(\mathcal{F}) + 1 + \tilde{\ell}(q) \geq \nu_q(C') + s_q(\mathcal{F}, p_q)$$

for every $q \in A$ and it is also clear that $D' \in H_{\tilde{\ell}}(a')$ and so $H_{\tilde{\ell}}(a') \neq \emptyset$. This means that C', $\tilde{\ell}$ and a' satisfy the hypothesis of the theorem and, taking $D' \in H_{\tilde{\ell}}(a')$, we are in the precedent case so one can conclude

$$\overset{\circ}{\partial} C - a + a' = \overset{\circ}{\partial} C' \leq \overset{\circ}{\partial} \mathcal{F} + 2 + a'$$

and the theorem is proved. □

Remark 5.4. The above theorem reduces the problem of bounding the degree of invariant curves of a foliation to the well known problem of imposing singularities on projective plane curves if we are able to give the function ℓ. This remark and the following one show two ways to use the theorem.

The first way is looking at the singularities of the foliation and trying to give a bound for the degree of the invariant curves in a certain class. For instance, if $S(\mathcal{F}) = \{p_1, \ldots, p_t\}$ and for each i, taking $D_i = \{q \in IN(p_i) / \varepsilon_q(\mathcal{F}) = 1\}$ and fixing functions $\phi_i : D_i \to \mathbb{N}$, then assume that at p_i the invariant curve C is controlled by ϕ_i and $\{B_q(\mathcal{F})\}_{q \in D_i}$ as in Proposition 4.5, for all $i \in \{1, \ldots, t\}$ (roughly speaking, if we limit the number of branches of C passing through each point with dicritical blowup). Then the theorem allows us to reduce the global problem of bounding the degree of C to the local one studied in the previous section. Thus, the proof of Proposition 4.5 tells us that, for all $q \in A = \bigcup_{i=1}^{t} IN(p_i)$, one has

$$\nu_q(\mathcal{F}) + 1 + \ell_q(\phi_{i(q)}, \mathcal{F}) \geq \nu_q(C) + s_q(\mathcal{F}, p_{i(q)})$$

where $i(q) \in \{1, \ldots, t\}$ is given by $q \in IN(p_{i(q)})$. If we take $\ell(q) = \ell_q(\phi_{i(q)}, \mathcal{F})$ the hypothesis of the theorem is fulfilled and we can compute a from ℓ (so we bound the degree of C).

For instance, given $p, q \in \mathbb{N}$ with $q < p$, $1 \neq q \neq p - 1$, take the foliation $\mathcal{F}_{p,q}$ given by the homogeneous differential form

$$\omega_{p,q} = p \cdot Z \cdot Y \cdot dX - q \cdot Z \cdot X \cdot dY + (q - p) \cdot Y \cdot X \cdot dZ.$$

Then $\mathcal{F}_{p,q}$ has three singular points, $p_1 = [1;0;0]$, $p_2 = [0;1;0]$ and $p_3 = [0;0;1]$, and $D_1 = \emptyset$ and D_2 and D_3 have exactly one point. Let q_i be the only point of D_i, $i = 2,3$, and take $\phi_2(q_2) = \phi_3(q_3) = n$. Since $1 \neq q \neq p-1$ we have $s_{q_i}(\mathcal{F}_{p,q}, p_i) = 2$ (and this simplification in the computation is the only reason to impose that condition) so for each $q \in IN(p_i)$ we have (see Proposition 4.5) $\ell(q) = \ell_q(\mathcal{F}_{p,q}, \phi_i) = \rho_q(q_i) \cdot n$, $i = 2,3$; in particular, $\ell(p_2) = (p-q) \cdot n$ and $\ell(p_3) = q \cdot n$. One can see that all the curves given by $\prod_{j=1}^{n}(X^p - \lambda_j \cdot Y^q \cdot Z^{p-q}) = 0$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, belong to $H_\ell(n \cdot p)$, so we can take $a = n \cdot p$. We obtain that $\partial^{\text{o}} C \leq n \cdot p + 3$ (because $\partial^{\text{o}} \mathcal{F}_{p,q} = 1$).

Since the foliation $\mathcal{F}_{p,q}$ is well known, we can check the kindness of the bound. The algebraic curves invariant by the foliation are the curves given by $\lambda \cdot X^p - \mu \cdot Y^q \cdot Z^{p-q}$, $\lambda, \mu \in \mathbb{C}$, and the three lines given by X, Y, and Z. The possible curves controlled by ϕ_2 at p_2 and by ϕ_3 at p_3 are the union of, at most, the three lines and n curves of degree p, so the bound can be realized.

Finally, in the general case with $S(\mathcal{F}) = \{p_1, \ldots, p_t\}$, if one has $D_i = \emptyset$ for $i = 1, \ldots, t$ (that is, if all the singularities are nondicritical) then one can take $\ell \equiv 0$ (Corollary 4.6) , a can be taken to be 0 and we recover the result of [5] as a particular case of the theorem.

Remark 5.5. The second way is predetermining the singularities of C. For instance, assume that $\{p_1, \ldots, p_t\}$ is the set of dicritical singularities of $\mathcal{F}$ (by the last remark we don't need to consider the nondicritical ones), and that respective equisingularity types (see [15]) $T_1, \ldots, T_t$ are fixed at the points $p_1, \ldots, p_t$. Let C be a reduced curve invariant by $\mathcal{F}$ and having equisingularity type T_i at p_i for every $i = 1, 2, \ldots, t$. For each $q \in A = \bigcup_{i=1}^{t} IN(p_i)$ let $i(q)$ be the element of $\{1, \ldots, t\}$ given by $q \in IN(p_{i(q)})$ and consider the function $\ell : A \to \mathbb{N}$ given by $\ell(q) = \nu_q(C) - 2 + e_q(p_{i(q)})$ if $\nu_q(C) \geq 1$ and $\ell(q) = 0$ if $\nu_q(C) = 0$.

Since C has an embedded resolution of singularities, it is clear that $\ell(q) = 0$ except for finitely many points q. Moreover, if $q \in C \cap IN(p_{i(q)})$ then $s_q(\mathcal{F}, p_{i(q)}) \neq 0$ implies $\nu_q(\mathcal{F}) > 0$ (because C is invariant by $\mathcal{F}$) so one has

$$1 + s_q(\mathcal{F}, p_{i(q)}) \leq e_q(p_{i(q)}) + \nu_q(\mathcal{F})$$

for every $q \in C \cap IN(p_{i(q)})$, and, hence

$$\nu_q(C) + s_q(\mathcal{F}, p_{i(q)}) \leq \nu_q(\mathcal{F}) + 1 + \ell(q)$$

for every $q \in A$ (it is easy to see that the inequality also holds when $\nu_q(C) = 0$). Now, since T_i is fixed, one can find an integer b_i, depending only on T_i, such that $\mathfrak{m}_{p_i}^{b_i} \subset \mathcal{I}_{\ell, p_i}$, $\mathfrak{m}_{p_i}$ being the maximal ideal of the local ring $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}, p_i}$. Thus, if $\ell^* : A \to \mathbb{N}$ is given by $\ell^*(p_i) = b_i$ and $\ell^*(q) = 0$ elsewhere, one has $\mathcal{I}_{\ell^*} \subset \mathcal{I}_\ell$ and if a is an integer such that the linear system $H_{\ell^*}(a)$ is nonempty, then one can conclude

$$\partial^{\text{o}} C \leq \partial^{\text{o}} \mathcal{F} + 2 + a.$$

Thus one gets a bound which depends only on $\mathcal{F}$ and the predetermined equisingularity types.

For concrete types of singularities the integers b_i are not difficult to compute. Thus, if T_i is an ordinary singularity of multiplicity at most n_i then $b_i = n_i - 2$. In particular, if C has only nodes as singularities one has $\ell = \ell^* \equiv 0$, a can be taken to be 0 and we recover the bound given in [7]. If T_i is a singularity of type A_{2k+1} (i.e., analytically isomorphic to $y^2 - x^{2k+1} = 0$), then, if $q_0 = p_i, q_1, \ldots$ denotes the successive points in $IN(p_i) \cap C$ (i.e., q_i is the only point in $C \cap IN(p_i)$ with $ht(q_i) = i$), one has $e_{q_j}(p_i) = 1$ for every j except $e_{q_0}(p_i) = 0$ and $e_{q_{k+1}}(p_i) = 2$, so one has $\ell(q_1) = \cdots = \ell(q_{k-1}) = \ell(q_{k+1}) = 1$ and $\ell(q_j) = 0$ elsewhere. By the discharge principle one finds $\ell'(q_0) = \ell'(q_1) = \cdots = \ell'(q_{k-1}) = 1$ and $\ell'(q_j) = 0$ for $j \geq k$ so one can take $b_i = k$. If T_i is of type A_{2k} (i.e., analytically isomorphic to $y^2 - x^{2k} = 0$) then as above one has $\ell(q_0) = 0$, $\ell(q_1) = \cdots = \ell(q_{k-1}) = 1$, $\ell(q_j) = 0$ for $j \geq k$; so, by the discharge principle, $\ell'(q_0) = \ell'(q_1) = \cdots = \ell'(q_{k-2}) = 1$ and one can take $b_i = k - 1$.

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